Physics Letters B 265 (1991) 124–126
North-Holland

PHYSICS LETTERS B

On the abelization theorem in the BRST approach

S.A. Frolov [1]

Steklov Mathematical Institute, Vavilov Street 42, GSP-1, SU-117 966 Moscow, USSR

Received 20 May 1991

The abelization theorem is proved on the quantum level in the framework of perturbation theory. An application of the theorem to the problem of the positivity of the physical subspace norm is considered.

The abelization theorem in classical mechanics states that one may locally abelize first-class constraints, i.e. make them commute among themselves by using some invertible matrix, depending on phase variables. The fact that one may abelize first-class constraints only locally means that in quantum mechanics this theorem may be generally valid only in the framework of perturbation theory. In this paper we shall give a formulation of the theorem in the BRST approach in a manner which is slightly different from the usual one and is more suitable to study the physical subspace, and then we shall prove the theorem in the framework of perturbation theory.

So, let us have a quantum nilpotent BRST operator Q, which may be represented in the following form:

$$Q=Q_0+gQ_1+...+\frac{1}{n!}g^nQ_n+... \qquad (1)$$

Here g is the coupling constant of the theory in question, and the operator Q_0 has the form

$$Q_0=\sum (c_i^+a_i^- + c_i^- a_i^+) . \qquad (2)$$

The operators $c^\pm$, $\bar{c}^\pm$, $a^\pm$, $\bar{a}^\pm$, are unphysical ones and have the following commutation and conjugation relations:

$$[c^-,\bar{c}^+]_+=[\bar{c}^-,c^+]_+$$
$$=[a^-,\bar{a}^+]_+=[\bar{a}^-,a^+]_+=1 ,$$
$$(c^-)^+=c^+, \quad (\bar{c}^-)^+=\bar{c}^+, \qquad (3)$$

$$(a^-)^+=a^+, \quad (\bar{a}^-)^+=\bar{a}^+ . \qquad (3\text{ cont'd})$$

The operator Q_0 is in fact the BRST operator of the linearized theory. As was shown in ref. [1] a BRST operator of any linearized theory in the hamiltonian BRST approach [2] may be reduced to the form (2). The operators $Q_1, ..., Q_n, ...$ depend on unphysical operators and on physical ones as well.

Now, the abelization theorem states that there exists some unitary operator U such that the following equality is valid:

$$U^+QU=Q_0 . \qquad (4)$$

To prove the theorem we will need an important statement concerning the cohomology of the operator Q_0. Namely, let an arbitrary operator f (anti)commute with the operator Q_0. Then, the operator f may be represented in the following form (see, for example, ref. [3]):

$$f=f_{ph}+[Q_0,\chi]_\mp , \qquad (5)$$

where f_{ph} depends only on the physical operators [#1].

Now we may prove the abelization theorem. To do that we will use a somewhat unusual form of the unitary operator U. We represent this operator as follows:

$$U-\frac{1+iK}{1-iK}, \qquad (6)$$

[1] E-mail address: frolov@qft.mian.su.

[#1] By using this theorem the existence of a quantum nilpotent BRST operator of the form (1) is not difficult to prove in the hamiltonian BRST approach.

Elsevier Science Publishers B.V.

where K is some hermitian operator.

Such a form of U is not suitable for practical applications but it simplifies the proof of the theorem, because in this case eq. (4) is quadratic with respect to the operator K:

$$Q-Q_0+K(Q-Q_0)K-\mathrm{i}[K,Q-Q_0]$$
$$=2\mathrm{i}[K,Q_0]\,. \qquad (7)$$

Let us represent the operator K as a perturbative series in the coupling constant g:

$$K=gK_1+\tfrac{1}{2}g^2K_2+...+\frac{1}{n!}g^nK_n+...\,, \qquad (8)$$

and rewrite eq. (7) in arbitrary order in g:

$$2\mathrm{i}[K_n,Q_0]=Q_n$$
$$+\sum_{m=2}^{n-1}\sum_{l=1}^{m-1}\frac{n!}{(n-m)!(m-l)!l!}K_{n-m}Q_{m-l}K_l$$
$$-\mathrm{i}\sum_{m=1}^{n-1}\frac{n!}{(n-m)!m!}[K_{n-m},Q_m]\,. \qquad (9)$$

We shall prove the existence of a solution of eq. (9) by means of complete induction. If $n=1$, then eq. (9) has the form

$$2\mathrm{i}[K_1,Q_0]=Q_1\,. \qquad (10)$$

The operator Q_1 anticommutes with the operator Q_0 due to the nilpotency of the BRST operator Q and, so, may be represented in the form $Q_1=[Q_0,\chi]$ (because there is no physical part of the operator Q_1). Thus, eq. (10) has a solution. Let us suggest that eq. (9) has solutions up to some order $n=N-1$. To prove the existence of a solution in the order $n=N$ we should show that the right-hand side of eq. (9) anticommutes with the free BRST operator Q_0. This may be done by direct inspection taking into account the nilpotency of the BRST operator Q.

So we see that the abelization theorem is a simple consequence of the nilpotency of the BRST operator and the triviality of the free BRST operator cohomology.

Finally, let us briefly discuss the action of the unitary operator U on the effective hamiltonian H:

$$H'=U^+HU\,. \qquad (11)$$

It is obvious that the transformed hamiltonian H' commutes with the free BRST operator Q_0, and so it may be represented in the following form due to the BRST cohomology theorem:

$$H'=H_{\rm ph}+[Q_0,\Psi]\,. \qquad (12)$$

The hamiltonian $H_{\rm ph}$ depends only on the physical variables and in fact coincides with the effective hamiltonian in some unitary gauge. This unitary gauge is defined by the unphysical operators $a^\pm$, $\bar{a}^\pm$. For example in the Lorentz gauge the usual choice of the operators $a^\pm$, $\bar{a}^\pm$ corresponds to the Coulomb gauge. Due to eq. (12) any matrix element of the evolution operator with the hamiltonian H is equal to the corresponding matrix element of the evolution operator with the hamiltonian $H_{\rm ph}$. Finally, let us notice that two arbitrary unitary gauges may be related to each other by some canonical transformation. Thus we have shown the equivalence between relativistic, unitary and temporal-like gauges (see ref. [4] about the BRST quantization of constrained systems in temporal-like gauges).

Now let us consider some simple examples of the application of the abelization theorem. Firstly, it is not difficult to see that any physical vector satisfying the condition $Q|\psi\rangle=0$ may be represented as follows:

$$|\Psi\rangle=U|\Psi\rangle_{\rm ph}+Q|\chi\rangle\,, \qquad (13)$$

where the vector $|\Psi\rangle_{\rm ph}$ depends only on the physical variables. Due to such a decomposition any physical vector has a positive norm which is equal to the norm of the vector $|\Psi\rangle_{\rm ph}$.

As the next example let us consider a BRST operator having the following form:

$$Q=\sum c_i^+(a_i^- -f_i)+\text{h.c.} \qquad (14)$$

Here f_i are arbitrary mutually commuting operators depending only on the physical variables. For example, the BRST operator of quantum electrodynamics has such a form. In this case the unitary operator U is equal to

$$U=\exp\left(\sum(\bar{a}_i^+f_i-\bar{a}_i^-f_i^*)\right). \qquad (15)$$

Although the physical subspace corresponding to the operator (14) has a perturbative structure, the eigenvalues of the hamiltonian H' may have a nonperturbative dependence on the coupling constant as for ex-

ample is the case in the Schwinger model (see ref. [5], where the Schwinger model was solved using the fermionic Fock space without bosonization).

I am grateful to A.A. Slavnov for valuable discussions.

References

[1] S.A. Frolov and A.A. Slavnov, Phys. Lett. B 218 (1989) 461.
[2] I.A. Batalin and E.S. Fradkin, Phys. Lett. B 122 (1983) 157; E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B 55 (1975) 224.
[3] M. Henneaux, Ann. Phys. 194 (1989) 281.
[4] S.A. Frolov, MIAN preprint No. 8 (1989); Theor. Math. Phys. 87 (1991) 188.
[5] R. Link, Phys. Rev. D 42 (1990) 2103.